Solution 1) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Then the characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda) - 2 \cdot 1\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-4) = 0$$

$\lambda = 1, 1, 4$ are the eigenvalues of A.

Let

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be the eigen vector corresponding to eigen value $\lambda = 1$.

Then $AX = X$

$$\Rightarrow (A - I)X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1]$$

Clearly the coefficient matrix of these equations is of rank 1, therefore these equations have $3 - 1 = 2$ L.I. solutions.

Now, (1) can be written as $\quad y + z = 0$

$$\Rightarrow y = -z$$

Taking $z = 1$, we get $y = -1$, $x = 1$

Taking $x = 1$, the eigen vector corresponding to eigen value 1 is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Again, let

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be the eigen vector corresponding to the eigen value $\lambda = 4$.

Then $AX = 4X$, i.e.

$$(A - 4I)X = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\text{Applying } R_3 \to R_3 + R_2]$$

These equations can be written as:

$$-3x + y + z = 0, \quad -2y + z = 0$$

Taking $z = 2$, we get $\quad x = 1,\ y = 1$

Hence $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue 4.

Solution 2) $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$$\text{Sum of Eigenvalues} = \text{Trace} = 3 + 2 + 5 = 10$$

$$\text{Product of Eigenvalues} = |A| = 30$$

Solution 3) Eigenvalues are $1, 2, 4$

$$\text{Sum of Eigenvalues} = \text{Trace}$$

$$1 + 2 + 4 = 2 + 3 + k \Rightarrow 7 = 5 + k \Rightarrow \boxed{k = 2}$$

Solution 4) Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$\text{Then } |A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$\begin{aligned}
&= (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[(\lambda-2) - (-1)\right] + 1\left[1 - (2-\lambda)\right] \\
&= (2-\lambda)\left[\lambda^2 - 4\lambda + 3\right] + (\lambda - 1) + (\lambda - 1) \\
&= (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-1) \\
&= (\lambda-1)[(2-\lambda)(\lambda-3) + 2] \\
&= (\lambda-1)(\lambda^2 - 5\lambda + 6 - 2) = (\lambda-1)(\lambda^2 - 5\lambda + 4) \\
&= -(\lambda-1)^2(\lambda-4)
\end{aligned}$$

The characteristic equation of A is $|A-\lambda I| = 0 \Rightarrow (\lambda-1)^2(\lambda-4) = 0 \Rightarrow \lambda = 1, 1, 4$

Eigenvalues of A are $1, 1, 4$

Eigenvalues of A^{-1} are $1, 1, \frac{1}{4}$

Solution 5) The characteristic equation of the given matrix is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)(9 - 6\lambda + \lambda^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda) = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

By trial, $\lambda = 2$ is a root of this equation.

$$\Rightarrow (\lambda-2)(\lambda^2 - 10\lambda + 16) = 0 \Rightarrow (\lambda-2)^2(\lambda-8) = 0$$

$\Rightarrow \lambda = 2, 2, 8$ are the characteristic roots or Eigen values.

Eigen values of A are $2, 2, 8$

Eigen values of $A^2 = (2)^2, (2)^2, (8)^2 = 4, 4, 64$

Solution 6)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

The eigenvalues of A are given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_3$:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 3 & 2-\lambda & -1 \\ 0 & 1 & 4-\lambda \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 0 & -2-\lambda & -3 \\ 0 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\begin{vmatrix} -2-\lambda & -3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \Rightarrow (3-\lambda)(\lambda-1)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 3, 3$$

Eigen values of $A = 1, 3, 3$

Eigen values of $A - I = 1-1, 3-1, 3-1 = 0, 2, 2$

Eigen values of $A + 5I = 1+5, 3+5, 3+5 = 6, 8, 8$

Solution 7)

$$A = \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix}$$

$$|A - \lambda I| = \left|\begin{bmatrix} 3-\lambda & -7 \\ 4 & 5-\lambda \end{bmatrix}\right| = (3-\lambda)(5-\lambda) + 28 = \lambda^2 - 8\lambda + 43$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - 8\lambda + 43 = 0$$

We are to show that A satisfies its characteristic equation i.e.

$$A^2 - 8A + 43I = 0$$

Now,

$$A^2 = AA = \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -19 & -56 \\ 32 & -3 \end{bmatrix}$$

$$\Rightarrow A^2 - 8A + 43I = \begin{bmatrix} -19 & -56 \\ 32 & -3 \end{bmatrix} - 8 \begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix} + 43 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -19 - 24 + 43 & -56 + 56 + 0 \\ 32 - 32 + 0 & -3 - 40 + 43 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{O}$$

i.e., $A^2 - 8A + 43I = \mathbb{O}$.
Hence Cayley-Hamilton theorem is verified.

Now, pre-multiplying both sides of (1) by A^{-1}, we get

$$A - 8I + 43A^{-1} = \mathbb{O} \Rightarrow 43A^{-1} = -A + 8I$$

$$= -\begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 + 8 & 7 + 0 \\ -4 + 0 & -5 + 8 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ -4 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{43} \begin{bmatrix} 5 & 7 \\ -4 & 3 \end{bmatrix}$$

Solution 8)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

The characteristic polynomial is the determinant of $A - \lambda I$, where I is the identity matrix and λ is the eigenvalue parameter.

First, we write $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & 2 - \lambda & 0 \\ 2 & -1 & -\lambda \end{bmatrix}$$

Now, compute the determinant of $A - \lambda I$:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & 2 - \lambda & 0 \\ 2 & -1 & -\lambda \end{vmatrix}$$

Expanding along the first row:

$$= (-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ -1 & -\lambda \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 2 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2-\lambda \\ 2 & -1 \end{vmatrix}$$

Thus, the characteristic polynomial of A is:

$$\lambda^3 - 2\lambda^2 - 2\lambda + 5 = 0$$

The Cayley-Hamilton theorem tells us that the matrix A satisfies its characteristic equation:

$$A^3 - 2A^2 - 2A + 5I = 0$$

Now, let's compute $A^2 = A \cdot A$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

Now, compute $A^3 = A^2 \cdot A$

$$A^3 = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & -2 & 2 \\ 6 & 7 & 2 \\ 2 & -6 & -1 \end{bmatrix}$$

We verify the Cayley-Hamilton theorem by substituting A^3, A^2, and A into the characteristic equation:

$$A^3 - 2A^2 - 2A + 5I = 0$$

Substituting the known values:

$$A^3 = \begin{bmatrix} -1 & -2 & 2 \\ 6 & 7 & 2 \\ 2 & -6 & -1 \end{bmatrix}, \quad 2A^2 = \begin{bmatrix} 4 & -2 & 0 \\ 4 & 8 & 2 \\ -2 & -4 & 4 \end{bmatrix}, \quad 2A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 4 & 0 \\ 4 & -2 & 0 \end{bmatrix}, \quad 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now subtract and add everything:

$$A^3 - 2A^2 - 2A + 5I = \begin{bmatrix} -1 & -2 & 2 \\ 6 & 7 & 2 \\ 2 & -6 & -1 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 0 \\ 4 & 8 & 2 \\ -2 & -4 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 2 & 4 & 0 \\ 4 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This confirms that the matrix satisfies the Cayley-Hamilton theorem!
From the Cayley-Hamilton theorem, we know:

$$A^3 - 2A^2 - 2A + 5I = 0 \quad \text{or} \quad A^3 = 2A^2 + 2A - 5I$$

Multiply both sides by A^{-1}:

$$A^2 = 2A + 2I - 5A^{-1}$$

Solving for A^{-1}:

$$A^{-1} = \frac{1}{5}\left(2A + 2I - A^2\right)$$

You can substitute the known values for A, A^2, and I to compute A^{-1}.

Solution 9)

Given:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

We want to express:

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

as a linear polynomial in A.
First, find the characteristic equation of A:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$$

So, the characteristic equation is:

$$\lambda^2 - 4\lambda - 5 = 0$$

By Cayley-Hamilton Theorem, A satisfies its own characteristic equation:

$$A^2 - 4A - 5I = 0 \quad (1) \Rightarrow A^2 = 4A + 5I$$

Multiply both sides of (1) by A, A^2, A^3, successively:

$$A^3 = 4A^2 + 5A \quad (2)$$
$$A^4 = 4A^3 + 5A^2 \quad (3)$$
$$A^5 = 4A^4 + 5A^3 \quad (4)$$

Now compute:

$$\begin{aligned}
A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I &= (4A^4 + 5A^3) - 4A^4 - 7A^3 + 11A^2 - A - 10I \quad \text{[Using (4)]} \\
&= -2A^3 + 11A^2 - A - 10I \\
&= -2(4A^2 + 5A) + 11A^2 - A - 10I \quad \text{[Using (2)]} \\
&= -8A^2 - 10A + 11A^2 - A - 10I \\
&= 3A^2 - 11A - 10I \\
&= 3(4A + 5I) - 11A - 10I \quad \text{[Using (1)]} \\
&= 12A + 15I - 11A - 10I \\
&= A + 5I
\end{aligned}$$

$$\boxed{A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I}$$

Solution 10)

Given:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Find characteristic equation:

Characteristic polynomial: $\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - 0\lambda - 2 = \lambda^2 - 2$

By Cayley-Hamilton Theorem:

$$A^2 - 2I = 0 \Rightarrow A^2 = 2I$$

Then:

$$A^4 = (A^2)^2 = (2I)^2 = 4I \Rightarrow A^8 = (A^4)^2 = 16I$$

So,

$$A^8 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

Now,

$$A^{10} = (A^2)^5 = (2I)^5 = 32I \Rightarrow A^{10} = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$$